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Noncommutativity in the mechanics of a free massless relativistic particle

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Abstract: We show the existence of a noncommutative spacetime structure in the context of a complete discussion on the underlying spacetime symmetries for the physical system of a free massless relativistic particle. The above spacetime symmetry transformations are discussed for the first-order Lagrangian of the system where the transformations on the coordinates, velocities and momenta play important roles. We discuss the dynamics of this system in a systematic manner by exploiting the symplectic structures associated with the four dimensional (non-)commutative cotangent (i.e. momentum phase) space corresponding to a two dimensional (non-)commutative configuration (i.e. target) space. A simple connection of the above noncommutativity (NC) is established with the NC associated with the subject of quantum groups where $SL_{q,q^{-1}}(2)$ transformations play a decisive role.

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1 Introduction

The noncommutative geometry and corresponding noncommutative field theories have generated a great deal of interest during the last few years due to their clear appearance in the context of string theories and their close cousins D-branes and M-theories (see, e.g., [1-4]). The end points of the open strings, trapped on the D-branes, turn out to be noncommutative in the presence of the antisymmetric ($B_{\mu\nu} = -B_{\nu\mu}$) potential $B_{\mu\nu}$ that constitutes the 2-form ($B = \frac{1}{2!}(dx^\mu \wedge dx^\nu) B_{\mu\nu}$) background field B. It has also been shown that, in a specific limit, the string dynamics can be described as a minimally coupled gauge field theory defined on a noncommutative space [2]. From a distinctly different perspective, the theoretical consideration of the quantum gravity and black hole physics entails upon the spacetime to become noncommutative in nature [5,6]. In other words, the NC is the benchmark of theoretical physics at an energy scale that is comparable to the Planck energy [6]. Physically, this NC amounts to the existence of an uncertainty relation between spacetime position operators which implies that the simultaneous measurement of the spacetime positions is not possible to a better accuracy than the Planck length (see, e.g., [5-10]).

Even though the NC is connected with the theoretical high energy physics applicable at the energy scale of the order of Planck energy, it is expected that the physical consequences of these NCs can be tested in the low energy effective actions for some physically interesting systems. This is why, to test the existence of such kind of NC, some experimental proposals have been made [11-13] where it has been argued that only the quantum mechanical effects are good enough to shed some light on the very existence of the NC in spacetime. There is an alternative possibility, however. One can construct the low energy theories by exploiting the basic ideas behind the NC in spacetime and can propose the physical consequences of these NCs on some experimentally testable physical quantities. On the theoretical side of the latter possibility, mention can be made of the noncommutative Chern-Simons theory [14] and the noncommutative standard model [15] where effects of the NC have been computed for some interesting physical quantities. However, experimentally, they are yet to be tested.

It is well-known that the physical insights and the spacetime symmetries behind the free and/or interacting relativistic (super) particle are at the heart of the modern developments in the understanding of the (super) string theories and (super) gravity theories. It is also well-understood that a massless scalar relativistic particle is endowed with more spacetime symmetries than its massive counterpart. In particular, the Lagrangian for the free massless relativistic particle respects the full conformal group of spacetime symmetries which is not the case for the free massive relativistic particle (cf. section 2). In a recent paper [16], it has been shown that the physical system of a massless relativistic particle possesses a new scale type of spacetime symmetry (in addition to the full conformal group of spacetime symmetries) which leads to an extension of the conformal algebra such that the NC in spacetime emerges naturally (see, e.g., [16] for details). Furthermore, in a recent couple of papers [17,18], the reparametrization symmetry and gauge symmetry of the *massive*

relativistic particle have been considered where the NC in spacetime has been shown to appear for the specific *choices* of the gauge in the framework of the Dirac bracket formalism. The system of massive and massless relativistic (super)particles have also been considered in the framework of quantum groups where the NC has been introduced in the cotangent (i.e. momentum phase) space of the above physical system [19-22].

In our present paper, we attempt to study the impact of the NC (in the spacetime structure) by laying emphasis on the dynamical aspects (i.e. equations of motion) as well as the spacetime symmetries associated with the free massless scalar relativistic particle. For this purpose, we exploit the symplectic structures in (i) the definition of the noncommutative Poisson brackets (and corresponding commutators), and (ii) the Legendre transformation to obtain the first-order Lagrangian. In the entire text, we focus on the symmetry properties of the first-order Lagrangian function together with the Euler-Lagrange equations of motion that emerge from this function. The main results of our present endeavour are in three folds. First, we attempt to look for the impact of the NC on the equations of motion of the free massless relativistic particle. Second, we exploit the mathematical sophistication of symplectic structures to study the details of the commutative and NC dynamics. Third, we establish a connection between (i) the NC of spacetime that emerges from the scale type of spacetime symmetry, and (ii) the NC of spacetime that originates from the quantum group $GL_{q,q^{-1}}(2)$ (and its special case $SL_{q,q^{-1}}(2)$) symmetry. It should be noted that all these results are *valid* only up to the lowest order (i.e. $\sim \beta = e^2 p^2$) in the parameters of the scale symmetry transformations that lead to the existence of NC in the theory.

In the context of the above results, it is pertinent to point out that, by thorough discussions on the key mathematical aspects of the dynamics, we find that the equations of motion remain unaffected due to the presence of the NC generated by the new scale type of spacetime symmetry in the theory. This observation is true to the lowest order in the parameters ($\sim \beta = e^2 p^2$) of the above scale type of spacetime symmetry transformation. It should be noted that, even though the NC exists in the commutators to the lowest order ($\sim \beta = e^2 p^2$, cf. (3.6)), the equations of motion (cf. (4.10), (4.32)) and the Hamiltonian (cf. footnote just after (4.32)) remain unaffected due to this NC because of the fact that all the terms, linear in the parameters ($\sim \beta = e^2 p^2$), cancel out. Furthermore, we have devoted a great deal of discussions on the derivation of the components of the covariant symplectic metric so that the equations of motion for the canonical *commutative* case as well as the nontrivial *noncommutative* case could be contrasted against each-other with utmost accuracy (cf. section 4). However, as it turns out, the equations of motion remain unchanged even though the parameters of NC (linear in $\sim \beta = e^2 p^2$) are present in the covariant symplectic structure (cf. (4.29)) on the noncommutative cotangent manifold.

We have been able to demonstrate a connection of the NC associated with the new scale type spacetime transformations to the NC associated with the quantum group $GL_{q,q^{-1}}(2)$ transformations on the phase variables where the elements of this group have been chosen in such a way that it automatically becomes $SL_{q,q^{-1}}(2)$. It turns out that the noncommutative

Poisson brackets between phase variables due to the $SL_{q,q^{-1}}(2)$ symmetry transformations reduce to the noncommutative Poisson brackets due to new scale type spacetime symmetry transformations for the deformation parameter satisfying $q^2 = 1$. Furthermore, the choice of the elements of $GL_{q,q^{-1}}(2)$ are such that the key q-algebraic relations among the phase variables [22] of the free massless relativistic particle remain intact on the cotangent manifold. To be more precise, for $q^2 = 1$, the relationships among the phase variables (that respect Lorentz invariance and $SL_{q,q^{-1}}(2)$ invariance) remain unchanged.

The contents of our present paper are organized as follows. In section 2, we set up the notations and conventions by discussing the bare essentials of the conformal, gauge- and reparametrization symmetry transformations for the Lagrangian of the free massless relativistic particle. For the paper to be self-contained, section 3 deals, in a somewhat different manner, the NC in the spacetime structure that owes its origin to an additional scale symmetry transformation [16]. Section 4 is devoted to the discussion of dynamics for the free massless relativistic particle in the four-dimensional (non-)commutative cotangent manifold. In section 5, we demonstrate a simple connection between the NC of spacetime due to an additional scale symmetry and the NC of spacetime due to quantum group $SL_{q,q^{-1}}(2)$ symmetry. Finally, we make some concluding remarks in section 6 and point out a few future directions for further investigations in the connected areas of research.

2 Preliminary: symmetries in Lagrangian formalism

We begin with the different looking but equivalent forms of the gauge- and reparametrization invariant Lagrangians for a free massive relativistic particle. The particle moves on a world-line (i.e. trajectory) that is embedded in a N-dimensional flat Euclidean target manifold †. The specific Lagrangians, describing the above massive particle, are [23,24]

$$L_0^{(m)} = m\,(\dot{x}^2)^{1/2} \qquad L_f^{(m)} = p_\mu \dot{x}^\mu - \tfrac{1}{2}\,e\,(p^2 - m^2) \qquad L_s^{(m)} = \tfrac{1}{2}\,e^{-1}\,(\dot{x})^2 + \tfrac{1}{2}\,e\,m^2. \qquad (2.1)$$

In the above, the mass-shell condition $(p^2 - m^2 = 0)$ and the force free (i.e. $\dot{p}_\mu = 0$) motion of the *free* massive relativistic particle are a couple of common features for (i) the Lagrangian with the square root $L_0^{(m)}$, (ii) the first-order Lagrangian $L_f^{(m)}$, and (iii) the second-order Lagrangian $L_s^{(m)}$. Except for the mass (i.e. the analogue of the cosmological constant) parameter m, the target space canonically conjugate coordinates $x^\mu(\tau)$ (with $\mu = 1, 2......N$) as well as the momenta $p_\mu(\tau)$ and the einbein field $e(\tau)$ are the functions of the monotonically increasing parameter τ that characterizes the trajectory (i.e. the world-line) of the free massive scalar relativistic particle. Here $\dot{x}^\mu = (dx^\mu/d\tau)$ are the generalized versions of the "velocity" of the particle. The first- and the second-order Lagrangians are endowed with the first-class constraints $\Pi_e \approx 0$ and $p^2 - m^2 \approx 0$ in the language of the

†For the Euclidean target manifold, we choose the metric $\delta_{\mu\nu} = \text{diag}\,(+1, +1, ...+1)$ and the scalar product between two vectors A_μ and B_μ is given by $(A \cdot B) = \delta_{\mu\nu} A_\mu B_\nu$. Thus, the contravariant vectors are same as the covariant vectors. In other words $A_\mu B^\mu = A_\mu B_\mu = A_1 B_1 + A_2 B_2 + ... + A_N B_N$. For the sake of convenience, however, we shall be using the upper and lower indices in the whole body of our text.

Dirac's classification scheme where Π_e is the canonical conjugate momentum corresponding to the einbein field $e(\tau)$. The existence of the first-class constraints on this physical system establishes the fact that this reparametrization invariant theory of the free massive relativistic particle is a *gauge* theory ‡. It is clear that the massless limit (i.e. $m \to 0$) is not consistently defined for the Lagrangian $L_0^{(m)}$ but the first- and the second-order Lagrangians do permit such a limit. The Lagrangians for the massless free relativistic particle, derived in such a limit (i.e. $m \to 0$), from $L_f^{(m)}$ and $L_s^{(m)}$:

$$L_f = p_\mu \dot{x}^\mu - \tfrac{1}{2}\, e\, p^2 \qquad\qquad L_s = \tfrac{1}{2}\, e^{-1}\, (\dot{x})^2 \tag{2.2}$$

are *not only* endowed with the following Poincaré (δ_p), reparametrization (δ_r) and gauge (δ_g) symmetry transformations which are also present for their massive counterparts (2.1) but they also respect scale (δ_s) and conformal (δ_c) symmetry transformations that are *not* present for the Lagrangians (2.1) for the massive free relativistic particle. In more sophisticated language, the breaking of the latter symmetries is said to generate the mass of the free particle. It is interesting to check that the second order Lagrangian L_s of (2.2) respects the following symmetry transformations

$$\begin{array}{lll}
\delta_g x_\mu = \xi (\dfrac{\dot{x}_\mu}{e}) & \delta_g e = \dot{\xi} & \delta_g L_s = \dfrac{d}{d\tau}\Big[\dfrac{\xi}{2}\, \dfrac{\dot{x}^2}{e^2}\Big] \\
\delta_s x_\mu = \alpha x_\mu & \delta_s e = 2 \alpha e & \delta_s L_s = 0 \\
\delta_c x_\mu = 2 x_\mu (x \cdot b) - b_\mu x^2 & \delta_c e = 4 e (x \cdot b) & \delta_c L_s = 0 \\
\delta_p x_\mu = \omega_\mu^\nu x_\nu + a_\mu & \delta_p e = 0 & \delta_p L_s = 0 \\
\delta_r x_\mu = \epsilon \dot{x}_\mu & \delta_r e = \dfrac{d}{d\tau}[\epsilon e] & \delta_r L_s = \dfrac{d}{d\tau}[\epsilon L_s]
\end{array} \tag{2.3}$$

where $\xi(\tau), \epsilon(\tau)$ are the *local* infinitesimal parameters corresponding to the gauge and reparametrization symmetry transformations, respectively, and $\omega^{\mu\nu}$ (with $\omega^{\mu\nu} = -\omega^{\nu\mu}$), a^μ, α and b^μ are the *global* infinitesimal parameters corresponding to the Poincaré transformations (ω, a), scale transformation (α) and conformal transformations (b^μ), respectively. The same symmetry transformations, for the first-order Lagrangian L_f, are

$$\begin{array}{llll}
\tilde{\delta}_g x_\mu = \xi p_\mu & \tilde{\delta}_g p_\mu = 0 & \tilde{\delta}_g e = \dot{\xi} & \tilde{\delta}_g L_f = \dfrac{d}{d\tau}\Big[\dfrac{\xi}{2}\, p^2\Big] \\
\tilde{\delta}_s x_\mu = \alpha x_\mu & \tilde{\delta}_s p_\mu = -\alpha p_\mu & \tilde{\delta}_s e = 2 \alpha e & \tilde{\delta}_s L_f = 0 \\
\tilde{\delta}_c x_\mu = 2 x_\mu (x \cdot b) - b_\mu x^2 & & \tilde{\delta}_c e \;=\; 4 e (x \cdot b) & \\
\tilde{\delta}_c p_\mu = 2 x_\mu (p \cdot b) - 2 b_\mu (x \cdot p) - 2 p_\mu (x \cdot b) & & & \tilde{\delta}_c L_f = 0 \\
\tilde{\delta}_p x_\mu = \omega_\mu^\nu x_\nu + a_\mu & \tilde{\delta}_p p_\mu = \omega_\mu^\nu p_\nu & \tilde{\delta}_p e = 0 & \tilde{\delta}_p L_f = 0 \\
\tilde{\delta}_r x_\mu = \epsilon \dot{x}_\mu & \tilde{\delta}_r p_\mu = \epsilon \dot{p}_\mu & \tilde{\delta}_r e = \dfrac{d}{d\tau}[\epsilon e] & \tilde{\delta}_r L_f = \dfrac{d}{d\tau}[\epsilon L_f].
\end{array} \tag{2.4}$$

‡The Lagrangian density $L_f^{(m)}$ transforms to $\delta_r L_f^{(m)} = (d/d\tau)[(\epsilon L_f^{(m)})]$ under the transformations $\delta_r x_\mu = \epsilon \dot{x}_\mu, \delta_r p_\mu = \epsilon \dot{p}_\mu, \delta_r e = (d/d\tau)[(\epsilon e)]$ generated by the basic reparametrization $\tau \to \tau - \epsilon(\tau)$ where $\epsilon(\tau)$ is an infinitesimal parameter. Similarly, under the gauge transformations $\delta_g x_\mu = \xi p_\mu, \delta_g p_\mu = 0, \delta_g e = \dot{\xi}$, the Lagrangian density $L_f^{(m)}$ transforms to a total derivative. Both these transformations are equivalent (with the identification $\xi = e\epsilon$) for the free (i.e. $\dot{p}_\mu = 0$) relativistic particle because both the above transformations owe their origin to the mass-shall condition $p^2 - m^2 = 0$. Thus, conditions $\dot{p}_\mu = 0$ and $p^2 - m^2 = 0$ are a couple of salient features for the dynamical description of the relativistic particle.

It will be noted that all the above symmetry transformations for the momentum variable p_μ, present in the first-order Lagrangian L_f, are such that there is a mutual consistency among the transformations on x_μ, e and p_μ so that the sanctity of the relation $p_\mu = e^{-1}\,\dot{x}_\mu$, derived from L_s or L_f, could be maintained. As explained earlier, the generator of the reparametrization transformation δ_r and the gauge transformation δ_g is the mass-shell condition $p^2 - m^2 = 0$ and $p^2 = 0$ for the massive- and massless cases, respectively. The operator form of the generators of all the above "conformal" transformations are

$$\begin{array}{ll} \hat{P}_\mu = \partial_\mu & \hat{K}_\mu = (2x_\mu x^\nu - x^2\delta^\nu_\mu)\,\partial_\nu \\ \hat{D} = x^\mu\partial_\mu & \hat{M}_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \end{array} \tag{2.5}$$

where the angular momentum operator $\hat{M}_{\mu\nu}$ and momentum operator $\hat{P}_\mu$ are the generators of the rotation and translation that constitute the full Poincaré transformations. The scale transformation δ_s is generated by the dilation operator $\hat{D}$ and the conformal transformation δ_c is generated by the conformal boost operator $\hat{K}_\mu$ (see, e.g., [16] for more details).

3 Additional symmetry and noncommutativity

It is clear from the starting Lagrangian $L_0^{(m)}$ in (2.1) that the Euler-Lagrange equations of motion are: $\ddot{x}_\mu(\dot{x})^2 - \dot{x}_\mu(\dot{x}\cdot\ddot{x}) = 0$. For the free motion $\ddot{x}^\mu = 0$ of the particle, we can choose a gauge such that $(\dot{x}\cdot\ddot{x}) = 0$ and $(\dot{x})^2 \neq 0$ (see, e.g., [23]). It is straightforward to notice that the mass m of the particle does not play any role at all in the dynamics of the particle. Thus, the first-order- and the second-order Lagrangians of (2.1) and (2.2) produce exactly the same type of equations of motion. These are encompassed in the free motion ($\dot{p}_\mu = 0$) of the particle and the definition of the canonical momentum (i.e. $p_\mu = e^{-1}\,\dot{x}_\mu$). In other words, the combination of these two relationships yields the following Euler-Lagrange equation of motion from the first- and the second-order Lagrangians

$$\ddot{x}_\mu e - \dot{x}_\mu\dot{e} = 0 \quad\Rightarrow\quad (\dot{x}\cdot\ddot{x})e - (\dot{x})^2\dot{e} = 0. \tag{3.1}$$

For the free motion $\ddot{x}_\mu = 0$ of the particle, we have to choose the following gauges: $(\dot{x}\cdot\ddot{x}) = 0$ and $\dot{e} = 0$ (but $e \neq 0$). There are a few compelling reasons for the above choices. First, in the limit ($e \longrightarrow 0$), we should recover the gauge choice imposed on the equations of motion derived from $L_0^{(m)}$ to obtain the free motion (i.e. $\ddot{x}_\mu = 0$). Second, the einbein field $e(\tau)$ is like the gauge field A_μ of the Abelian 1-form gauge theory where the Lorentz gauge $(\partial\cdot A) = 0$ is just the analogue of $\dot{e} = 0$. In fact, this gauge choice is exploited in the Becchi-Rouet-Stora-Tyutin (BRST) quantization (see, e.g., [21,24]) of the free massive as well as massless relativistic particle [§]. Third, it is evident that the constraint $p^2 \approx 0$ is

[§]In fact, the (anti-)BRST invariant Lagrangian $L_b = p_\mu\dot{x}^\mu - \frac{1}{2}e(p^2 - m^2) + b\dot{e} + \frac{1}{2}b^2 - i\dot{\bar{c}}\dot{c}$ does exploit the gauge-fixing term $\dot{e}$ through the Nakanishi-Lautrup auxiliary field b. In terms of the (anti-)ghost fields $(\bar{c})c$, the nilpotent $(s^2_{(a)b} = 0)$ (anti-)BRST symmetry transformations $s_{(a)b}$ are: $s_b x_\mu = c p_\mu, s_b p_\mu = 0, s_b c = 0, s_b e = \dot{c}, s_b\bar{c} = ib, s_b b = 0$ and $s_{ab}x_\mu = \bar{c}p_\mu, s_{ab}p_\mu = 0, s_{ab}\bar{c} = 0, s_{ab}e = \dot{\bar{c}}, s_{ab}c = -ib, s_{ab}b = 0$ [21,24].

the generator of the gauge- and reparametrization symmetries (cf. (2.4)) for the massless relativistic particle. Exploiting $p_\mu = e^{-1}\dot{x}_\mu$, it can be seen that the time invariance of the above constraint $(d/d\tau)[\, p^2\,] = 0$ (which is equivalent to $(p \cdot \dot{p}) = 0$) implies that the gauge choices $(\dot{x} \cdot \ddot{x}) = 0$ and $\dot{e} = 0$ are consistent with the free motion $\ddot{x}_\mu = 0$ of the massless relativistic particle. Utilizing the free motion $\dot{p}_\mu = 0$ and the above gauge choices (i.e. $\dot{e} = 0$ and $(\dot{x} \cdot \ddot{x}) = 0$), it is straightforward to check that the following scale transformations on the spacetime coordinate x_μ, momentum p_μ and einbein field $e(\tau)$

$$\begin{array}{lcl} x_\mu(\tau) \to X_\mu(\tau) & = & e^{\beta(\dot{x}^2)}\, x_\mu(\tau) \equiv e^{\beta(e^2 p^2)}\, x_\mu(\tau) \\ p_\mu(\tau) \to P_\mu(\tau) & = & e^{-\beta(\dot{x}^2)}\, p_\mu(\tau) \equiv e^{-\beta(e^2 p^2)}\, p_\mu(\tau) \\ e(\tau) \to E(\tau) & = & e^{2\beta(\dot{x}^2)}\, e(\tau) \equiv e^{2\beta(e^2 p^2)}\, e(\tau) \end{array} \tag{3.2}$$

leave the first-order Lagrangian L_f invariant primarily because of the fact that the above transformations imply: $\dot{x}^\mu \to \dot{X}^\mu = e^{\beta(e^2p^2)}\dot{x}^\mu$. This is an extension, albeit in a restricted sense, of the conformal symmetries listed in (2.3) or (2.4). As a consequence, the dilatation operator $\hat{D} = x^\mu \partial_\mu$ is extended to $\hat{D}^* = (1 + \beta)\hat{D}$. Furthermore, the above new symmetry allows an extension of the conformal algebra which has been discussed thoroughly in [16]. The infinitesimal version (i.e. $\beta^n \approx 0, n \geq 2$) of (3.2), which are of physical importance in spacetime symmetries, are

$$\begin{array}{lcl} x_\mu(\tau) \to X_\mu(\tau) & = & x_\mu(\tau) + \beta(e^2p^2)\, x_\mu(\tau) \\ p_\mu(\tau) \to P_\mu(\tau) & = & p_\mu(\tau) - \beta(e^2p^2)\, p_\mu(\tau) \\ e(\tau) \to E(\tau) & = & e(\tau) + 2\, \beta(e^2p^2)\, e(\tau). \end{array} \tag{3.3}$$

It is now clear that the following canonical brackets

$$[x_\mu, x_\nu] = 0 \quad [p_\mu, p_\nu] = 0 \quad [x_\mu, p_\nu] = i\delta_{\mu\nu} \quad [x_\mu, e] = 0 \quad [p_\mu, e] = 0 \tag{3.4}$$

emerging from the first- and the second-order Lagrangians of the massless relativistic particle, are now changed to their noncommutative counterparts as

$$\begin{array}{l} [X_\mu(\tau), X_\nu(\tau)] \;=\; (1+\beta)\, \Big\{[x_\mu(\tau), \beta]\, x_\nu(\tau) + [\beta, x_\nu(\tau)]\, x_\mu(\tau)\Big\} \\ [X_\mu(\tau), P_\nu(\tau)] \;=\; (1+\beta)\, \Big\{i\delta_{\mu\nu}\, (1-\beta) \;-\; [x_\mu(\tau), \beta]\, p_\nu(\tau)\Big\} \\ [P_\mu(\tau), E(\tau)] = 0 \qquad\qquad\quad [E(\tau)(\tau), E(\tau)] = 0 \\ [X_\mu(\tau), E(\tau)] \;=\; 2\, (1+\beta)\, [x_\mu(\tau), \beta]\, e(\tau) \qquad [P_\mu(\tau), P_\nu(\tau)] = 0. \end{array} \tag{3.5}$$

A few comments, at this juncture, are in order now. First, it is obvious from the above that the limit $\beta \to 0$ produces the canonical brackets (3.4) for the massless free relativistic particle. Second, the commutator between the transformed momentum fields as well as the einbein fields remains the same as their untransformed canonical form. Third, the spacetime becomes noncommutative very naturally due to the new symmetry transformations (3.2) (and its infinitesimal version(3.3)). This should be contrasted with the NC that emerges due to the choices of the gauge in the context of the Dirac bracket formalism (see, e.g., [17,18]). Fourth, it will be noted that the brackets in (3.5) are still up to the order β^2.

However, for the discussion of the dynamics in the next section, we shall be concentrating only on the contributions coming from the brackets of order β. Finally, for one of the simplest choices: $\beta(e^2p^2) = e^2p^2$, the above brackets (3.5) become

$$\begin{array}{lll} [X_\mu(\tau), X_\nu(\tau)] &=& -2ie^2(1+e^2p^2)\Big\{(x_\mu(\tau)p_\nu(\tau) - x_\nu(\tau)p_\mu(\tau)\Big\} \equiv -2ie^2 M^*_{\mu\nu} \\ [X_\mu(\tau), P_\nu(\tau)] &=& (1+e^2p^2)\Big\{i\delta_{\mu\nu}\,(1-e^2p^2) - 2ie^2 p_\mu\, p_\nu\Big\} \\ [P_\mu(\tau), E(\tau)] = 0 & & [E(\tau), E(\tau)] = 0 \\ [X_\mu(\tau), E(\tau)] &=& 4\, i\, e^3\, (1+e^2p^2)\, p_\mu \qquad [P_\mu(\tau), P_\nu(\tau)] = 0 \end{array} \tag{3.6}$$

where $M^*_{\mu\nu} = (1+e^2p^2)M_{\mu\nu}$. This shows that (i) the NC of the spacetime in the above owes its origin to the rotation in the off-shell (i.e. $p^2 \neq 0$) reference frame that is generated by the (off-shell) angular momentum operator $M^*_{\mu\nu} = (1+\beta)M_{\mu\nu}$ where $\beta = e^2p^2$. (ii) The antisymmetric property of the commutator (i.e. $[X_\mu, X_\nu] = -[X_\nu, X_\mu]$) is encoded in the antisymmetric property of the angular momentum operator (i.e. $M^*_{\mu\nu} = -M^*_{\nu\mu}$). (iii) It is evident that one of the key requirements [10] of the NC (i.e. $\int Tr[X_\mu, X_\nu] = 0$) of spacetime geometry is fulfilled, in the above, due to the antisymmetric property of $M^*_{\mu\nu}$ (see, e.g., [10]). (iv) It is interesting to check that the brackets in (3.5) and (3.6) do satisfy all the possible Jacobi identities among the phase space variables (see, e.g., [16]).

4 Symplectic structures and dynamics

For the sake of simplicity, we shall focus on the motion of the free massless relativistic particle on a two-dimensional Euclidean target (i.e. configuration) space parametrized by the coordinate variables $x_1(\tau)$ and $x_2(\tau)$. The corresponding four-dimensional phase (i.e. cotangent) space is parametrized by the four variables $x_1(\tau), x_2(\tau), p_1(\tau), p_2(\tau)$ where $p_\mu(\tau)$'s (with $\mu = 1, 2$) are the canonical conjugate momenta corresponding to the coordinate variables $x_\mu(\tau)$'s ($\mu = 1, 2$). The following Hamiltonian function $H(1)$

$$H(1) = \frac{1}{2}\, e(\tau)\, p^2(\tau) \equiv \frac{1}{2}\, e(\tau)\, [\, p_1^2(\tau) + p_2^2(\tau)\,] \tag{4.1}$$

with the canonical commutators $[x_\mu(\tau), p_\nu(\tau)] = i\delta_{\mu\nu}, [x_\mu(\tau), e(\tau)] = 0, [p_\mu(\tau), p_\nu(\tau)] = 0, [p_\mu(\tau), e(\tau)] = 0, [x_\mu(\tau), x_\nu(\tau)] = 0$, leads to the following equations of motion

$$\dot{x}_\mu(\tau) = -\, i\, [\, x_\mu(\tau), H(1)\,] = e(\tau)\, p_\mu(\tau) \qquad \dot{p}_\mu(\tau) = -\, i\, [\, p_\mu(\tau), H(1)\,] = 0 \tag{4.2}$$

which imply the validity of (3.1) as well as the free motion (i.e. $\dot{p}_\mu = 0$) of the massless relativistic particle. Classically, the above canonical commutators correspond to the canonical Poisson brackets $\{x_\mu, x_\nu\}_{(PB)} = 0, \{p_\mu, p_\nu\}_{(PB)} = 0, \{x_\mu, p_\nu\}_{(PB)} = \delta_{\mu\nu}$ on the four dimensional symplectic (i.e. cotangent) manifold with the following contravariant and covariant symplectic structures

$$\Omega^{AB}(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \Omega_{AB}(1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.3}$$

where "1" in the round brackets after Ω^{AB} and Ω_{AB} stands for similar round bracket in the Hamiltonian function (4.1) and the notation $z^A = (z^1, z^2, z^3, z^4) \equiv (x_1, x_2, p_1, p_2)$ has been introduced in the definition of the matrix form of the symplectic structures as

$$\Omega^{AB} = \text{Matrix}(\{z^A, z^B\}_{(PB)}) \qquad \Omega^{AB}\Omega_{BC} = \delta^A_C = \Omega_{CM}\Omega^{MA}. \tag{4.4}$$

The most general form of the Poisson brackets between any two arbitrary dynamical variables $F(z)$ and $G(z)$ on the symplectic cotangent manifold is defined by exploiting the contravariant symplectic structure as

$$\{F(z), G(z)\}_{(PB)} = \Omega^{AB}\, \partial_A F(z) \partial_B G(z) \qquad \partial_A = \frac{\partial}{\partial z^A}. \tag{4.5}$$

On the other hand, the covariant symplectic structure plays a pivotal role in the definition of the Legendre transformation which leads to the derivation of the first-order Lagrangian from a given Hamiltonian. In general, the symplectic structures can be functions of the phase variables z^A. In such a case, the general form of the Legendre transformation is given by (see, e.g., [25,26] for details)

$$L_f(z, \dot{z}) = z^A \bar{\Omega}_{AB}(z) \dot{z}^B - H(z) \tag{4.6}$$

where the general form of the covariant symplectic structure $\bar{\Omega}_{AB}(z)$ is [25,26]

$$\bar{\Omega}_{AB}(z) = \int_0^1 d\kappa\; \kappa\; \Omega_{AB}(\kappa z). \tag{4.7}$$

For our present case of the symplectic structures, defined in (4.3) and satisfying (4.4), the above formula yields $\bar{\Omega}^{AB}(1) = \frac{1}{2}\, \Omega_{AB}(1)$. Modulo some total derivatives with respect to τ, the above equation (4.6) for the Legendre transformation, produces the following first order Lagrangian

$$L_f(1) = p_\mu(\tau) \dot{x}^\mu(\tau) - \tfrac{1}{2}\, e(\tau) p^2(\tau) \tag{4.8}$$

which is exactly the same as in (2.2) for the massless relativistic particle. The equations of motion from the above Lagrangian lead to the derivation of exactly the same equations of motion as given in (3.1) and the substitution $p_\mu = e^{-1} \dot{x}_\mu$ in (4.8) produces L_s of (2.2).

We shall now exploit the above simple discussion in the context of the noncommutative commutators given in (3.5) (and its special case given in (3.6)) for the choice $\beta = e^2 p^2$. First of all, using the infinitesimal transformations in (3.3), we obtain the transformed version of the Hamiltonian function $H(2)$ from its untransformed version $H(1)$, as

$$H(1) = \tfrac{1}{2}\, e(\tau)\; p^2(\tau) \quad \Rightarrow \quad H(2) = \tfrac{1}{2}\, E(\tau)\; P^2(\tau). \tag{4.9}$$

It is interesting to point out that the *form* of our beginning equations of motion, derived from $H(1)$ and expressed in terms of the untransformed variables, remains *unchanged* up to the lowest order in $\sim e^2 p^2$ when the equations of motion, in terms of the transformed

variables, are derived from the transformed Hamiltonian $H(2)$. This can be checked by considering the following Heisenberg's equations of motion for the transformed variables

$$\begin{aligned} \dot{X}_\mu(\tau) &= -\, i\,[\, X_\mu(\tau), H(2)\,] \equiv -\tfrac{i}{2}\,[X_\mu(\tau), E(\tau)]\; P^2(\tau) \\ &- \tfrac{i}{2}\, E(\tau)\; [X_\mu(\tau), P^2(\tau)] \qquad \dot{P}_\mu(\tau) = -\, i\,[\, P_\mu(\tau), H(2)\,] = 0 \end{aligned} \tag{4.10}$$

where we have exploited the commutation relations of (3.6) to prove the free motion ($\dot{P}_\mu = 0$) of the particle. Furthermore, with the help of (3.6), it can be seen that the explicit expressions for the commutators of (4.10), up to the order $\beta = e^2 p^2$, are

$$\begin{aligned} &-\tfrac{i}{2}\,[X_\mu(\tau), E(\tau)]\; P^2(\tau) \approx 2\,(e p_\mu)\,(e^2 p^2) + \mathcal{O}(e^4 p^4) + ... \\ &-\tfrac{i}{2}\, E(\tau)\,[X_\mu(\tau), P^2(\tau)] \approx E(\tau) P_\mu(\tau) - 2\,(e p_\mu)\,(e^2 p^2) + \mathcal{O}(e^4 p^4). \end{aligned} \tag{4.11}$$

This demonstrates that, using the non-trivial commutators of (3.6), we obtain the equations of motion from the transformed Hamiltonian $H(2)$

$$\dot{X}_\mu(\tau) = E(\tau)\; P_\mu(\tau) \qquad \dot{P}_\mu(\tau) = 0 \tag{4.12}$$

which dynamically corresponds to the same equations of motion as $\dot{x}_\mu = e p_\mu, \dot{p}_\mu = 0$. It should be noted that the equations of motion remain *form invariant* only up to the order $\beta = e^2 p^2$ of the transformations in (3.2). They do not retain this form-invariance even at the next order (i.e. $\beta^2 = e^4 p^4$). To obtain the analogue of (4.8) for the noncommutative brackets (cf. (3.6)), we have to exploit the analogues of the definitions in (4.6) and (4.7) which heavily depend on the explicit form of the symplectic structures. To this end in mind, we obtain the Poisson brackets from (3.6) (valid up to the order $\sim e^2 p^2$) as

$$\begin{aligned} \{X_\mu(\tau), X_\nu(\tau)\}_{(PB)} &= -2\, e^2 (1 + e^2 p^2) \Big(x_\mu\, p_\nu - x_\nu\, p_\mu \Big) \\ \{X_\mu(\tau), P_\nu(\tau)\}_{(PB)} &= \delta_{\mu\nu}\; - 2 e^2 (1 + e^2 p^2)\; p_\mu\, p_\nu \quad \{P_\mu(\tau), P_\nu(\tau)\}_{(PB)} = 0. \end{aligned} \tag{4.13}$$

In more explicit form, the above brackets yield the following brackets for the noncommutative four-dimensional cotangent manifold parametrized by four phase variables ¶

$$\begin{aligned} &\{X_1, X_1\}_{(PB)} = 0 \qquad \{X_2, X_2\}_{(PB)} = 0 \\ &\{X_1, X_2\}_{(PB)} = J_{12}(z) \qquad \{X_2, X_1\}_{(PB)} = -J_{12}(z) \\ &J_{12}(z) = -2\, e^2\, (1\; +\; e^2\, p^2)\, (x_1\, p_2 - x_2\, p_1) \end{aligned} \tag{4.14}$$

$$\begin{aligned} &\{X_1, P_1\}_{(PB)} = 1 - 2 e^2 (1 + e^2 p^2)\; p_1^2 \equiv S_{11}(z) \\ &\{X_2, P_2\}_{(PB)} = 1 - 2 e^2 (1 + e^2 p^2)\; p_2^2 \equiv S_{22}(z) \\ &\{X_1, P_2\}_{(PB)} = -2 e^2 (1 + e^2 p^2)\; p_1 p_2 \equiv S_{12}(z) \\ &\{X_2, P_1\}_{(PB)} = -2 e^2 (1 + e^2 p^2)\; p_2 p_1 \equiv S_{12}(z) \\ &\{P_1, P_1\}_{(PB)} = 0 \qquad \{P_2, P_2\}_{(PB)} = 0 \\ &\{P_1, P_2\}_{(PB)} = 0 \qquad \{P_2, P_1\}_{(PB)} = 0 \end{aligned} \tag{4.15}$$

¶It should be noted that an actual computation of the Poisson brackets between spacetime variables X_μ's is : $\{X_\mu(\tau), X_\nu(\tau)\}_{(PB)} = +2\; e^2 (1 + e^2 p^2)\; (p_\mu x_\nu - p_\nu x_\mu)$. However, to be consistent with the corresponding commutator in (3.6), we have exploited the substitution $(p_\mu x_\nu - p_\nu x_\mu) = -(x_\mu p_\nu - x_\nu p_\mu)$ which, to be very precise, is valid for the usual definition of a commutator $[x_\mu p_\nu] = x_\mu p_\nu - p_\nu x_\mu = i \delta_{\mu\nu}$. We have followed here the usual convention that connects a commutator with the corresponding Poisson bracket (i.e. $[F, G] = i \{F, G\}_{(PB)}$) for a couple of dynamical variables F and G.

where now the symbol z^A stands for: $z^A = (z^1, z^2, z^3, z^4) = (X_1, X_2, P_1, P_2)$. According to our definition in (4.4), we obtain the following contravariant symplectic structure

$$\Omega^{AB}_{(2)}(z) = \begin{pmatrix} 0 & J_{12}(z) & S_{11}(z) & S_{12}(z) \\ -J_{12}(z) & 0 & S_{12}(z) & S_{22}(z) \\ -S_{11}(z) & -S_{12}(z) & 0 & 0 \\ -S_{12}(z) & -S_{22}(z) & 0 & 0 \end{pmatrix}. \tag{4.16}$$

The covariant symplectic structure, corresponding to the above contravariant symplectic structure and satisfying (4.4), is

$$\Omega^{(2)}_{AB}(z) = \frac{1}{S^2_{12}(z) - S_{11}(z)S_{22}(z)} \begin{pmatrix} 0 & 0 & S_{22}(z) & -S_{12}(z) \\ 0 & 0 & -S_{12}(z) & S_{11}(z) \\ -S_{22}(z) & S_{12}(z) & 0 & -J_{12}(z) \\ S_{12}(z) & -S_{11}(z) & J_{12}(z) & 0 \end{pmatrix}. \tag{4.17}$$

Exploiting the definition (4.7), we compute the covariant symplectic metric $\bar{\Omega}^{(2)}_{AB}(z)$ which will be useful in the context of the Legendre transformations (4.6). It is elementary to check that $\bar{\Omega}^{(2)}_{ij}(z) = 0$ for $i, j = 1, 2$. The next non-trivial component of the covariant symplectic metric is

$$\bar{\Omega}^{(2)}_{13}(z) = \int_0^1 d\kappa \; \kappa \; \Big[\; \frac{S_{22}(\kappa z)}{S^2_{12}(\kappa z) - S_{11}(\kappa z) S_{22}(\kappa z)} \; \Big] \tag{4.18}$$

where by explicit computation, it can be seen that, to the order $\sim e^2 p^2$, we have

$$\begin{array}{ll} S_{22}(\kappa z) \approx (1 - 2e^2 p_2^2 \kappa^2) \qquad & S^2_{12}(\kappa z) \approx 4 \; (e^4 p_1^2 p_2^2) \; \kappa^4 \approx 0 \\ S_{11}(\kappa z) \approx (1 - 2e^2 p_1^2 \kappa^2) \qquad & S_{11}(\kappa z) S_{22}(\kappa z) \approx (1 - 2e^2 p^2 \kappa^2). \end{array} \tag{4.19}$$

Insertions of these values in (4.18) yield

$$\bar{\Omega}^{(2)}_{13}(z) = -\int_0^1 d\kappa \; \kappa \; \Big[\; \frac{(1 - 2e^2 p_2^2 \kappa^2)}{(1 - 2e^2 p^2 \kappa^2)} \; \Big] \approx -\frac{1}{2} (\; 1 + e^2 p^2 \;) \tag{4.20}$$

where we have used the following integrals

$$\begin{array}{l} -\int_0^1 \Big[\; \frac{d\kappa \; \kappa}{(1 - 2e^2 p^2 \kappa^2)} \; \Big] = \frac{1}{4e^2 p^2} \int_1^{1 - 2e^2 p^2} \frac{dt}{t} \; \equiv -\frac{1}{2} \; (1 + e^2 p^2) + \mathcal{O}(e^4 p^4) + ... \\ +2e^2 p_2^2 \int_0^1 \Big[\; \frac{d\kappa \; \kappa^3}{(1 - 2e^2 p^2 \kappa^2)} \; \Big] = -\frac{p_2^2}{4e^2 p^2} \int_1^{1 - 2e^2 p^2} \Big[\; \frac{dt}{t} - dt \; \Big] \equiv 0 + \mathcal{O}(e^4 p^4). \end{array} \tag{4.21}$$

Thus, it is now evident that, up to the order $\sim e^2 p^2$, we have

$$\bar{\Omega}^{(2)}_{13}(z) = \bar{\Omega}^{(2)}_{24}(z) = -\tfrac{1}{2} \; (1 + e^2 p^2) \qquad \bar{\Omega}^{(2)}_{31}(z) = \bar{\Omega}^{(2)}_{42}(z) = +\tfrac{1}{2} \; (1 + e^2 p^2). \tag{4.22}$$

Another interesting component of the covariant symplectic metric is

$$\bar{\Omega}^{(2)}_{14}(z) = -\int_0^1 d\kappa \; \kappa \; \Big[\; \frac{S_{12}(\kappa z)}{S^2_{12}(\kappa z) - S_{11}(\kappa z) S_{22}(\kappa z)} \; \Big] \approx +\int_0^1 d\kappa \; \kappa \; \Big[\; \frac{S_{12}(\kappa z)}{(1 - 2e^2 p^2 \kappa^2)} \; \Big] \tag{4.23}$$

where, up to the order $\sim e^2p^2$, $S_{12}(\kappa z) \approx -2e^2 p_1 p_2 \kappa^2$. Ultimately, the above integral, with the substitution $1 - 2e^2p^2\kappa^2 = t$, reduces to

$$\bar{\Omega}^{(2)}_{14}(z) = -2e^2 p_1 p_2 \int_0^1 d\kappa \; \Big[\; \frac{\kappa^3}{(1 - 2e^2p^2\kappa^2)} \; \Big] \approx \frac{p_1 p_2}{2e^2p^4} \int_1^{1-2e^2p^2} \; \Big[\; \frac{dt}{t} \; - dt \; \Big] \tag{4.24}$$

which yields the value of the integral equal to *zero* up to the order $\sim e^2p^2$. With this result, it is straightforward to see that $\bar{\Omega}^{(2)}_{14}(z) = \bar{\Omega}^{(2)}_{41}(z) = 0, \; \bar{\Omega}^{(2)}_{23}(z) = \bar{\Omega}^{(2)}_{32}(z) = 0$. Only one more useful computation is left over as far as the complete derivation of the covariant metric $\bar{\Omega}^{(2)}_{AB}(z)$ is concerned. This is as follows

$$\bar{\Omega}^{(2)}_{34}(z) = -\int_0^1 d\kappa \; \kappa \; \Big[\; \frac{J_{12}(\kappa z)}{S^2_{12}(\kappa z) - S_{11}(\kappa z) S_{22}(\kappa z)} \; \Big] \approx +\int_0^1 d\kappa \; \kappa \; \Big[\; \frac{J_{12}(\kappa z)}{(1 - 2e^2p^2\kappa^2)} \; \Big] \tag{4.25}$$

where, up to the order $\sim e^2p^2$, we have the following form for the $J_{12}(\kappa z)$, namely;

$$J_{12}(\kappa z) \approx -2e^2(x_1 p_2 - x_2 p_1) \; \kappa^2 - 2e^2(e^2p^2)(x_1 p_2 - x_2 p_1) \; \kappa^4. \tag{4.26}$$

Thus, the integral (4.25) finally looks in the following form

$$\begin{aligned} \bar{\Omega}^{(2)}_{34}(z) \;\; &= \;\; -2e^2(x_1 p_2 - x_2 p_1) \int_0^1 d\kappa \; \Big[\; \frac{\kappa^3}{(1 - 2e^2p^2\kappa^2)} \; \Big] \\ &- \;\; 2e^2(e^2p^2) \; (x_1 p_2 - x_2 p_1) \int_0^1 d\kappa \; \Big[\; \frac{\kappa^5}{(1 - 2e^2p^2\kappa^2)} \; \Big]. \end{aligned} \tag{4.27}$$

From our earlier discussions, it is clear that the first integral will be zero up to the order $\sim e^2p^2$. The second integral, with the substitution $1 - 2e^2p^2\kappa^2 = t$, becomes

$$\frac{(x_1 p_2 - x_2 p_1)}{2p^2} \int_1^{1-2e^2p^2} \; \Big[\; \frac{dt}{t} \; - 2dt + t \; dt \Big] \tag{4.28}$$

which is equal to zero up to the order $\sim e^2p^2$. This finally implies that, up to the order $\sim e^2p^2$, we have: $\bar{\Omega}^{(2)}_{34}(z) = \bar{\Omega}^{(2)}_{43}(z) = 0$. With the help of the above inputs, the covariant symplectic metric $\bar{\Omega}^{(2)}_{AB}(z)$, useful for the Legendre transformation (4.6), becomes

$$\bar{\Omega}^{(2)}_{AB}(z) = \frac{1}{2} \begin{pmatrix} 0 & 0 & -(1 + e^2p^2) & 0 \\ 0 & 0 & 0 & -(1 + e^2p^2) \\ 1 + e^2p^2 & 0 & 0 & 0 \\ 0 & 1 + e^2p^2 & 0 & 0 \end{pmatrix}. \tag{4.29}$$

Exploiting the above covariant symplectic metric, we obtain the first-order Lagrangian $L_f(2)$, using equation (4.6), as given below

$$\begin{aligned} L_f(2) \;\; &= \;\; \tfrac{1}{2} \, p_1 \; (1 + e^2p^2) \; \dot{x}_1 + \tfrac{1}{2} \, p_2 \; (1 + e^2p^2) \; \dot{x}_2 - \tfrac{1}{2} \, x_1 \; (1 + e^2p^2) \; \dot{p}_1 \\ &- \;\; \tfrac{1}{2} \, x_2 \; (1 + e^2p^2) \; \dot{p}_2 - \tfrac{1}{2} \; e \; (p_1^2 + p_2^2) \end{aligned} \tag{4.30}$$

where we have used the fact that, up to the order $\sim e^2p^2$, the Hamiltonian functions $H(1)$ and $H(2)$ are one and the same. Now the stage is set to recall the dynamical restrictions

of section 3 where we have imposed $\dot{e} = 0, (\dot{x} \cdot \ddot{x}) = 0$ (which is also equivalent to $\dot{e} = 0, (p \cdot \dot{p}) = 0$). Tapping this information, the above first-order Lagrangian (with $\dot{X}_1 = (1 + e^2 p^2)\, \dot{x}_1, \dot{X}_2 = (1 + e^2 p^2)\, \dot{x}_2, X_\mu = (1 + e^2 p^2) x_\mu \equiv x_\mu (1 + e^2 p^2)$) can be recast into

$$\tilde{L}_f(2) = \tfrac{1}{2}\, p_1\, \dot{X}_1 + \tfrac{1}{2}\, p_2\, \dot{X}_2 - \tfrac{1}{2}\, X_1\, \dot{p}_1 - \tfrac{1}{2}\, X_2\, \dot{p}_2 - \tfrac{1}{2}\, e\, (p_1^2 + p_2^2) \tag{4.31}$$

which leads to the derivation of the following Euler-Lagrange equations of motion

$$\dot{X}_\mu(\tau) = e p_\mu \qquad\qquad \dot{p}_\mu = 0. \tag{4.32}$$

The above equations are same as the equations of motion (4.12) (with a bit changed notations). Thus, the first-order Lagrangian that emerges from the metric (4.29), is same as (4.8) (modulo the fact that x_μ's are now replaced by X_μ) and therefore the form of the equations of motion $\dot{x}_\mu = e p_\mu$ and $\dot{p}_\mu = 0$ remain unchanged up to the order $\sim e^2 p^2$ as they can be also written as $\dot{X}_\mu = E P_\mu$ and $\dot{P}_\mu = 0$. In other words, the dynamics ‖ remains unchanged up to the order $\sim e^2 p^2$. This feature is exactly same as our earlier discussion on the Landau problem where, despite the presence of the NC, the equations of motion for the charged particle under the influence of the magnetic field, remain unchanged [27].

5 Connection with quantum groups

First of all, let us recapitulate some of the pertinent points of our earlier work [22] related to the construction of a consistent dynamics on a four dimensional noncommutative cotangent manifold. In this connection, it can be checked that (i) the ordinary Lorentz invariance, and (ii) a particular (i.e. $pq = 1$) quantum group $GL_{q,p}(2)$ invariance are respected together for any arbitrary ordering of the indices μ, ν (with $\mu, \nu = 1, 2$) in the following relationship between the phase variables on the cotangent manifold [22]

$$x_\mu x_\nu = x_\nu x_\mu \qquad p_\mu p_\nu = p_\nu p_\mu \qquad x_\mu p_\nu = q\; p_\nu x_\mu. \tag{5.1}$$

The phase variables $x_\mu (\mu = 1, 2)$ and $p_\mu (\mu = 1, 2)$, in the above, undergo the following change under the quantum group $GL_{q,p}(2)$ transformations

$$\begin{array}{ccc} \begin{pmatrix} x_1 \\ p_1 \end{pmatrix} \to \begin{pmatrix} X_1 \\ P_1 \end{pmatrix} & = & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \end{pmatrix} \\ \begin{pmatrix} x_2 \\ p_2 \end{pmatrix} \to \begin{pmatrix} X_2 \\ P_2 \end{pmatrix} & = & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_2 \\ p_2 \end{pmatrix} \end{array} \tag{5.2}$$

where the elements A, B, C, D of the 2×2 quantum matrix belonging to the quantum group $GL_{q,p}(2)$ obey the braiding relations in rows and columns as

$$\begin{array}{llll} AB = pBA & AC = qCA & BD = qDB & BC = (q/p) CB \\ CD = pDC & AD - DA \;=\; (p - q^{-1})\, BC = (q - p^{-1})\, CB. & & \end{array} \tag{5.3}$$

‖In particular, it can be checked explicitly that $H(2) = \frac{1}{2} E P^2 \to H(1) = \frac{1}{2} e p^2$ under transformations (3.3) up to the order $\sim e^2 p^2$. Thus, despite the presence of NC in (3.6), the dynamics remains unchanged.

It will be noted that the noncommutative algebraic relations (5.1) remain form-invariant (i.e. $X_\mu X_\nu = X_\nu X_\mu, P_\mu P_\nu = P_\nu P_\mu, X_\mu P_\nu = qP_\nu X_\mu$) under (5.2) only for the non-zero complex deformation parameters q, p (i.e. $q, p \in \mathcal{C}/\{0\}$) satisfying the restriction : $pq = 1$. In other words, the quantum group $GL_{q,q^{-1}}(2)$ is responsible for the form-invariance of the relations (5.1) on the cotangent manifold. For this group, the q-algebraic relations (5.3) among the elements A, B, C, D reduce to the following simpler form

$$\begin{array}{ccc} AB = q^{-1}BA & AC = qCA & BD = qDB \\ BC = q^2CB & CD = q^{-1}DC & AD = DA. \end{array} \tag{5.4}$$

At this stage, two comments are in order. First, for our present discussion about the dynamics, we have chosen the four dimensional Euclidean noncommutative cotangent manifold *only* for the sake of simplicity. Our present discussions, however, can be generalized to any arbitrary $2N$-dimensional ($N > 2$) cotangent manifold in a straightforward manner. Second, it will be noted that the elements A, B, C, D of the quantum group $GL_{q,q^{-1}}(2)$ are assumed to *commute* (i.e. $Ax_\mu = x_\mu A, p_\mu A = Ap_\mu, AX_\mu = X_\mu A$, etc.) with the phase variables (x_μ, p_μ) and (X_μ, P_μ) in the proof of the form-invariance of (5.1).

Now the stage is set for a thorough discussion on the new scale transformations for the phase variables: $x_\mu \to X_\mu = (1 + e^2p^2)x_\mu, p_\mu \to P_\mu = (1 - e^2p^2)p_\mu$ of equation (3.3) for the choice $\beta = e^2p^2$ in the framework of the quantum groups. To this end in mind, it can be seen that for $B = 0, C = 0$ in (5.2) (i) we obtain a scale type of transformations for the phase variables (i.e. $x_\mu \to X_\mu = Ax_\mu, p_\mu \to P_\mu = Dp_\mu$), (ii) the noncommutative relations of (5.1) remain form-invariant for $B = C = 0$, and (iii) the algebraic relations (5.4) reduce to a single relationship $AD = DA$. In all the above observations, we have assumed the *commutativity* of the elements A, D with the phase variables x_μ and p_μ. This crucial assumption must be *maintained* in our attempt to capture the transformations (3.3) for $\beta = e^2p^2$ in the framework of the quantum groups. For instance, for the choice $A = (1 + e^2p^2)$ and $D = (1 - e^2p^2)$, it can be seen that $AD = DA = 1$ up to the order $\sim e^2p^2$. Thus, for the above choice of A and D (with $B = C = 0$), we see that (i) the relationship $AD = DA$, corresponding to the quantum group $GL_{q,q^{-1}}(2)$ is satisfied, and (ii) the scale transformations $X_\mu = (1 + e^2p^2)\ x_\mu$ and $P_\mu = (1 - e^2p^2)\ p_\mu$ of (3.3) (i.e. $\beta = e^2p^2$) are captured in the language of the quantum group transformations. Thus, two of the above three basic requirements are readily fulfilled. In addition, the condition $AD = DA = 1$ enforces the quantum group $GL_{q,q^{-1}}(2)$ to reduce to the quantum group $SL_{q,q^{-1}}(2)$ because the determinant of the matrix becomes *one* (i.e. $AD = 1$). Finally, let us now concentrate on the form-invariance (i.e. $X_\mu X_\nu = X_\nu X_\mu, P_\mu P_\nu = P_\nu P_\mu, X_\mu P_\nu = qP_\nu X_\mu$) of the relationships in (5.1) for the choice $A = (1+e^2p^2)$ and $D = (1-e^2p^2)$. In this context, it is pertinent to recall that the following noncommutative q-algebraic relations [22]

$$\begin{array}{ccccc} \dot{x}_\mu x_\nu = x_\nu \dot{x}_\mu & \dot{x}_\mu \dot{x}_\nu = \dot{x}_\nu \dot{x}_\mu & \dot{x}_\mu p_\nu = qp_\nu \dot{x}_\mu & p_\mu p_\nu = p_\nu p_\mu & \\ e\dot{x}_\mu = q\dot{x}_\mu e & ep_\mu = qp_\mu e & ex_\mu = qx_\mu e & x_\mu x_\nu = x_\nu x_\mu & x_\mu p_\nu = qp_\nu x_\mu \end{array} \tag{5.5}$$

are valid for the massless q-deformed free relativistic particle described by the Lagrangian $L = \frac{1}{2}q^{-1}e^{-1}\dot{x}^2$ and the Hamiltonian $H = \frac{1}{2}ep^2$ [22]. The above noncommutative q-

algebraic relations have been derived from the consistent differential calculi developed on the $GL_{q,q^{-1}}(2)$ invariant quantum hyperplane. It can be checked that the form-invariance of (5.1) could be maintained for our choice $A = (1 + e^2 p^2)$ and $D = (1 - e^2 p^2)$ *only* for the restriction $q^2 = 1$. In fact, this latter restriction emerges from the requirement $X_\mu P_\nu = q P_\nu X_\mu$ when we exploit the relations (5.5) that lead to

$$p_\mu D = (1 - q^{-2} e^2 p^2)\, p_\mu \qquad\qquad p_\mu A = (1 + q^{-2} e^2 p^2)\, p_\mu. \tag{5.6}$$

Basically, in the above, the restriction on the deformation parameter (q) emerges because of our demand that the *commutativity* requirement of our earlier discussion (i.e. $p_\mu D = D p_\mu, p_\mu A = A p_\mu$) should be true even if A and D are chosen to be explicitly dependent on the phase variables. Such a requirement is very much essential because the derivation of all the q-algebraic relations in (5.5), the definition of the q-deformed Poisson bracket (see, eqn. (5.7) below), etc., are based on the assumption that the elements of the $GL_{q,q^{-1}}(2)$ group commute with the physical phase variables x_μ and p_μ. It should be emphasized that the restriction $q^2 = 1$ has also appeared in the context of Landau problem [27] and the requirement of the equivalence between the gauge symmetry and reparametrization symmetry for the case of a q-deformed relativistic (super)particle [19-21]. It is interesting to point out that the relations $X_\mu X_\nu = X_\nu X_\mu, P_\mu P_\nu = P_\nu P_\mu$ remain sacrosanct for the choice $A = (1 + e^2 p^2), D = (1 - e^2 p^2)$ for any arbitrary value of q if we use the noncommutative algebraic relations (5.5). This happens primarily due to the natural commutativity $x_\mu A = A x_\mu, x_\mu D = D x_\mu$ of x_μ with A and D as well as the commutativity $(1 - e^2 p^2)(1 - q^{-2} e^2 p^2) = (1 - q^{-2} e^2 p^2)(1 - e^2 p^2)$. The restriction $q^2 = 1$ also emerges from the equality of the Poisson brackets (4.13) with the Poisson brackets calculated from the consideration of the quantum group $GL_{q,q^{-1}}(2)$. The definition of a Poisson bracket between two dynamical variables $F(x,p)$ and $G(x,p)$ on the $GL_{q,q^{-1}}(2)$ invariant cotangent manifold, is [22]

$$\big\{F, G\big\}^{(q)}_{PB} = \frac{\partial G}{\partial p_\lambda}\, \frac{\partial F}{\partial x_\lambda} - q\, \frac{\partial G}{\partial x_\lambda}\, \frac{\partial F}{\partial p_\lambda} \tag{5.7}$$

where (i) the repeated index is assumed to be summed over (i.e. $\lambda = 1, 2$), and (ii) the key ingredients have been taken from the differential calculi developed on the $GL_{q,q^{-1}}(2)$ invariant quantum hyperplane. It will be noted that, in the above, the derivatives are defined as the "left" derivatives. This amounts to bringing all the specific variables to the left by exploiting the q-algebraic relations of (5.5) before the differentiation w.r.t. that specific variable could be carried out. Exploiting the above definition, it can be seen that the following q-deformed noncommutative Poisson brackets, analogous to (4.13), emerge in the framework of the quantum group $GL_{q,q^{-1}}(2)$ and corresponding differential calculi developed on the $GL_{q,q^{-1}}(2)$ invariant hyperplane, namely;

$$\begin{aligned}
\{X_\mu(\tau), X_\nu(\tau)\}^{(q)}_{(PB)} &= +2q^2\, e^2\, [\, (1 + q^{-2} e^2 p^2)\, p_\mu\, x_\nu \\
&- q\, (1 + q^{-4} e^2 p^2)\, p_\nu\, x_\mu\,] \\
\{X_\mu(\tau), P_\nu(\tau)\}^{(q)}_{(PB)} &= [1 + (1 - q^2) e^2 p^2]\, \delta_{\mu\nu} - 2q^2 e^2 [1 + q^{-4} e^2 p^2]\, p_\mu\, p_\nu \\
\{P_\mu(\tau), P_\nu(\tau)\}^{(q)}_{(PB)} &= 0.
\end{aligned} \tag{5.8}$$

A few comments, at this juncture, are in order. First, it can be seen that in the limit $q \to 1$, we get back our Poisson brackets (4.13). Second, the key restriction on the deformation parameter q, that emerges due to the equality between (4.13) and (5.8) is, once again, $q^2 = 1$. This is due to the fact that an extra q-factor that appears in the definition of the q-Poisson bracket in (5.7) as well as in (5.8) (see the second term on the r.h.s. of the q-bracket $\{X_\mu, X_\nu\}^{(q)}_{PB}$) is due to the choice of the contravariant symplectic metric (see, e.g., [22]) **. Third, to shed some more light on the left-derivative, it can be seen that the explicit form of a specific q-Poisson bracket is

$$\begin{aligned} \Big\{X_\mu(\tau), X_\nu(\tau)\Big\}^{(q)}_{(PB)} &= \frac{\partial}{\partial p_\lambda}[(1 + e^2p^2)\, x_\nu]\, \frac{\partial}{\partial x_\lambda}[(1 + e^2p^2)\, x_\mu] \\ &- q\, \frac{\partial}{\partial x_\lambda}[(1 + e^2p^2)\, x_\nu]\, \frac{\partial}{\partial p_\lambda}[(1 + e^2p^2)\, x_\mu]. \end{aligned} \tag{5.9}$$

The meaning of the "left-derivative" in the differentiation $(\partial/\partial p_\lambda)[e^2p^2]$ is the trick that, using the q-algebraic relations of equation (5.5), the variable p^2 should be brought to the left. It can be checked that $e^2p^2 = q^4p^2e^2$ so that p^2 is reordered to the left. Now, we apply the left derivative on it. This operation yields $2q^4p_\lambda e^2$. This can be further rearranged to yield $2q^2e^2p_\lambda$. This differentiation is carried out by exploiting the differential calculi developed in [22]. All the Poisson brackets for the phase variables in (5.8) have been computed by exploiting the above trick.

6 Conclusions

In our present investigation, we have demonstrated the existence of the noncommutative spacetime structure in the context of a thorough discussion on the spacetime symmetry properties of the physical system of a free massless scalar relativistic particle. The presence of an additional (i.e. new) scale type of spacetime symmetry transformation for this system entails upon the spacetime to become noncommutative in nature. This new scale type of spacetime symmetry is drastically different from the usual scale type of symmetry that belongs to the usual set of conformal group of spacetime symmetry transformations. As a consequence, the usual conformal algebra gets modified and the NC in the spacetime geometry arises through the noncommutative algebraic structure [16].

It is worthwhile to compare and contrast the usual scale spacetime symmetry and the additional scale spacetime symmetry. The key differences are (i) the usual type of the scale spacetime symmetry is a *global* symmetry (cf. (2.3) and (2.4)) but the new scale spacetime symmetry is a *local* one *per se.* (ii) The dependence of the local parameter β of the additional new scale type of spacetime symmetry is very specific (i.e. $\beta(\dot{x}^2) = \beta(e^2p^2)$) whereas

**The q-deformed Poisson bracket, expressed in (5.7), can be defined in a more symmetric fashion: $\{F, G\}^{(q)}_{(PB)} = q^{-1/2}(\partial G/\partial p_\lambda)(\partial F/\partial x_\lambda) - q^{+1/2}(\partial G/\partial x_\lambda)(\partial F/\partial p_\lambda)$. This expression corresponds to the choice of a contravariant symplectic metric that differs from the one chosen in [22] by a constant factor $q^{1/2}$. In this form of the definition of the Poisson bracket, the restriction $q^2 = 1$ becomes more transparent.

the parameter α of the usual scale spacetime transformation is global (i.e. spacetime independent). (iii) It is the requirement of the free ($\ddot{x}_\mu = 0, \dot{p}_\mu = 0$) motion of the massless relativistic particle that enforces the choice of the gauges $\dot{e} = 0, (\dot{x} \cdot \ddot{x}) = 0$ (or equivalently $\dot{e} = 0, (p \cdot \dot{p}) = 0$) which ultimately turn out to be responsible for the existence of the new scale type spacetime symmetry (cf. (3.2) and (3.3)). There is no such type of criterion for the existence of the usual scale type of spacetime symmetry (cf. (2.3) and (2.4)). (iv) It is the requirement of the consistency and complementarity between the dynamics and the spacetime symmetries that are at the heart of the existence of the NC in spacetime structure in our present investigation. This NC is *intrinsically* different from the NC that arises in the context of the massive relativistic particle where the choice of gauges leads to the NC in the spacetime structure in the Dirac bracket formalism [17,18].

One of the central ingredients in our whole discussion is to focus on the impact of the NC on the dynamics of the free massless scalar relativistic particle. In particular, the discussions on the equations of motion for this system have been given utmost priority. As it turns out, the NC of the spacetime does not affect the equations of motion up to the order ($\sim e^2 p^2$). This feature is exactly same as our earlier work on the Landau problem [27] where the classical equations of motion remain unaffected by the presence of the NC in the theory. In our present paper, we have carried out a detailed and systematic computation for the derivation of the equations of motion from the Lagrangian and Hamiltonian formulations. The key role in all these discussions is played by the contravariant- and covariant symplectic structures which turn out to be responsible for (i) the systematic definition of the Poisson brackets on the cotangent (momentum phase) space, and (ii) the consistent definition of the Legendre transformations, respectively. In particular, section 4 of our present paper is devoted to a thorough discussion of dynamics where we have taken into account the NC of the Poisson brackets and corresponding symplectic structures. However, despite the presence of the parameters of NC in the covariant symplectic metric (cf. (4.29)), the dynamics remains unchanged up to the lowest order in the parameter of the NC.

We have attempted to establish a connection between the NC of spacetime due to the presence of a new scale type of spacetime symmetry and the NC of spacetime due to the presence of a quantum group $GL_{q,q^{-1}}(2)$ type symmetry for which a consistent dynamics has been developed in [22]. In our present investigation, the $2N$-dimensional differential calculi and the dynamics of [22] have been reduced to the differential calculi for a four dimensional cotangent (momentum phase) space where the ordinary Lorentz invariance and $GL_{q,q^{-1}}(2)$ invariance are respected together for any arbitrary ordering of the Lorentz indices. The key point, in the above connection between two types of NC, is the choice of the elements (i.e. $B = C = 0, A = (1 + e^2 p^2), D = (1 - e^2 p^2)$) of the 2×2 quantum matrix belonging to $GL_{q,q^{-1}}(2)$ which entails upon (i) the spacetime to become noncommutative in nature (see, e.g., eqns. (5.8), (5.9)), (ii) the quantum group transformations to capture the new scale transformations (3.3) for $\beta = e^2 p^2$, and (iii) the quantum group symmetry transformations to correspond to the $SL_{q,q^{-1}}(2)$ transformations because the determinant

of the above 2×2 quantum matrix becomes one (i.e. $AD = DA = 1$ for $B = C = 0$). It should be noted, however, that the transformations on the einbein field $e(\tau)$ (cf. (3.3)) are not captured by the quantum group transformations. This is why the analogues of the equations of motion (4.10) and (4.32) have not been discussed in the framework of the quantum group $SL_{q,q^{-1}}(2)$. Thus, our consideration of the connection between the NC due to the new scale spacetime symmetry and the NC due to the quantum group $SL_{q,q^{-1}}(2)$ symmetry is confined only to the analogy between the Poisson brackets (4.13) and (5.8). As it turns out, in the limit $q^2 = 1$, the NC in the spacetime due to the quantum group $SL_{q,q^{-1}}(2)$ symmetry transformations is reduced to the NC of the spacetime due to the new type of scale symmetry transformations for the system of a free massless relativistic particle.

It would be interesting endeavour to extend our present work to the case of a massless spinning relativistic (super)particle where the reparametrization and supersymmetric transformations co-exist. It is expected that the consideration of this system under the super quantum group $GL_{\sqrt{q}}(1|1)$ [20] might turn out, at some stage, to be quite handy. Furthermore, the noncommutative realization of the cohomological operators for the super quantum group $GL_{q,q^{-1}}(1|1)$ [28,29] also might play some crucial roles in this context. We hope to apply our present work to the case of a massive relativistic particle where some kinds of gauge transformations and reparametrization transformations have been shown to be connected with one-another under a general scheme [17]. These are some of the issues that are under investigation and our results would be reported elsewhere [30].

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References

[1] Witten E 1990 *Nucl. Phys.* B **460** 33 (1990)

[2] Seiberg N and Witten E 1999 *J. High Energy Phys.* **9909** 032

[3] Connes A, Douglas M R and Schwarz A 1998 *J. High Energy Phys.* **9802** 033

[4] For a review, see, e.g., Castellani L 2000 *Class. Quant. Grav.* **17** 3377

[5] Doplicher S, Fredenhagen K and Roberts J 1994 *Phys. Lett.* B **331** 39
Doplicher S, Fredenhagen K and Roberts J 1995 *Comm. Math. Phys.* **172** 187

[6] Ahluwalia D V 1994 *Phys. Lett.* B **339** 301 (*Preprint* gr-qc/9308007)
Compean H G, Obregon O and Ramirez C 2002 *Phys. Rev. Lett.* **88** 161301

[7] Jackiw R 2002 *Nucl. Phys. Proc. Suppl.* **108** 30 (*Preprint* hep-th/0110057)

[8] Snyder H S 1947 *Phys. Rev.* **71** 38
Yang C N 1947 *Phys. Rev.* **72** 874

[9] Yukawa H 1953 *Phys. Rev.* **53** 415
For a recent review, see, e.g., Tanaka S 2003 From Yukawa to M-theory (*Preprint* hep-th/0306047)

[10] Madore J 1999 *An introduction to noncommutative differential geometry and its physical applications* (Cambridge University Press: Cambridge)

[11] Castorina P, Iorio A and Zappalà D 2004 *Phys. Rev.* D **69** 065008

[12] Falomir H, Gamboa J, Loewe M, Mendes F and Rojas J C 2000 *Phys. Rev.* D **66** 045018

[13] Chaichian M, Demichev A, Presnajder P, Sheikh-Jabbari M M and Tureanu A 2002 *Phys. Lett.* B **527** 149

[14] L. Susskind 2001 The quantum Hall fluid and noncommutative Chern-Simons theory (*Preprint* hep-th/0101029)
Polychronakos A P 2001 *J. High Energy Phys.* **0104** 011

[15] Calmet X, Jurco B, Schupp P, Wess J and Wohlgenannt M 2002 *Euro. Phys. J.* C **23** 363

[16] Chagas-Filho W 2004 Extended conformal algebra and noncommutative geometry in particle theory (*Preprint* hep-th/0406071)

[17] Banerjee R, Chakraborty B and Gangopadhyay S 2004 Reparametrization symmetry and noncommutativity in particle mechanics (*Preprint* hep-th/0405178)

[18] Pinzul A and Stern A 2004 Space-time noncommutativity from particle mechanics (*Preprint* hep-th/0402220)

[19] Malik R P 1993 *Phys. Lett.* B **316** 257 (*Postprint* hep-th/0303071)

[20] Malik R P 1995 *Phys. Lett.* B **345** 131 (*Preprint* hep-th/9410173)

[21] Malik R P 1996 *Mod. Phys. Lett.* A **11** 2871 (*Preprint* hep-th/9503025)

[22] Malik R P, Mishra A K and Rajasekaran G 1998 *Int. J. Mod. Phys.* A **13** 4759 (*Preprint* hep-th/9707004)

[23] Brink L, Deser S, Zumino B, Di Vecchia P and Howe P 1976 *Phys. Lett.* B **64** 435
Brink L, Di Vecchia D and Howe P 1977 *Nucl. Phys.* B **183** 76

[24] Nemschansky D, Preitschopf C and Weinstein M 1988 *Ann. Phys.* (N Y) **183** 226

[25] Batalin I A and Fradkin E S 1989 *Nucl. Phys.* B **326** 701

[26] Shabanov S V 1993 *J. Phys. A: Math. Gen.* **26** 2583

[27] Malik R P 2003 *Mod. Phys. Lett.* A **18** 2795 (*Preprint* hep-th/0302224)

[28] Malik R P 2004 *J. Phys. A: Math. Gen.* **37** 8383 (*Preprint* hep-th/0309127)

[29] Isaev A P and Malik R P 1992 *Phys. Lett.* B **280** 219 (*Postprint* hep-th/0309186)

[30] Malik R P, in preparation.